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TRANSFORMATION FOR STATISTICAL DISTRIBUTION APPROXIMATELY NORMAL BUT OF FINITE SAMPLE RANGE

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San Diego, California

October 1967

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Work was performed under SR 104 03 01, Task 0586 (NEL L40571). The report covers work from August 1965 to October 1966 under NEL auspices, having been initiated earlier under other auspices. The report was approved for publication 30 October 1967.

The author is Professor of Statistics at the University of Connecticut. He performed the study as a part-time member of the Oceanometrics Group. The work was partially supported by a Public Health Service Research Grant GM 11608 from the National Institute of General Medical Sciences.

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THE PROBLEM

Develop statistical, physical, and computer techniques and methods for interpreting, summarizing, and extrapolating environmental data to support Navy requirements in research, developmental, and operational aspects of underwater detection, location, communications, and navigation. Specifically, develop a statistical test applicable to data samples drawn from populations that are approximately normal but have bounded, rather than infinite, domains.

RESULTS

Two traditional approaches are in use for hypothesis testing of data samples drawn from populations that are approximately normal but have finite domains: (a) assumption of a distribution that closely approximates the sample, such as a Beta distribution or one of the Pearson curves, and (b) truncation (removing the tails) of an assumed normal population. Approach (a) leads to intractable mathematics, and it is difficult to justify the appropriate curve and parameters. Approach (b) leads to serious errors in hypothesis testing if the frequency curve approaches zero at the bounds of the domain, since the tails of the assumed distribution are used in computing probabilities of error and separating decision regions.

In this study, a transformation is performed on an assumed truncated normal distribution so that many of the desirable statistical properties of normal distributions are retained, yet the data sample distribution in the tails is more closely matched than is possible through use of traditional approaches.

The major results of the study are:

1. Development of sampling theory and tests of hypothesis for

   a. mean tests on samples with known variance.

   b. mean tests on samples with estimated variance.

   c. tests of variance.

2. Application of the technique to two data samples, one concerned with a problem in electronic component reliability, the other with ocean temperature analysis.

3. Preparation of extensive tables to permit practical use of the method.

RECOMMENDATIONS

1. In application of statistical tests to specific problems, examine the satisfaction of assumptions when using normal tests such as t-tests or F-tests. When the assumptions are violated by domain restrictions, use the method described in this study.

2. Study the use of a sometimes more realistic Beta or Pearson distribution as an alternative approach to the problem, and compare results so attained with results obtained using the approach presented in this report.

INTRODUCTION

In many cases data are drawn from populations with approximately normal distributions but with bounded, rather than infinite, domain. Suppose the population density for a variate x is denoted $g_c^*(x; 0, 1)$, $x \in [-c, c]$, and the associated standard normal is $f(x; 0, 1)$. (The 0, 1 imply zero mean and unit standard deviation.) The traditional way of treating this problem is to assume a truncated normal distribution, for which the theory is well known. Suppose this truncated normal is $f_c^{(t)}(x)$. However, if $g_c^*(x) \to 0$ as $x \to c^-$ (or $x \to -c^+$), then

$$f_c^{(t)}(x) / g_c^*(x) \text{ grows large as } x \to c^- \text{ (or } x \to -c^+\text{)} \tag{1}$$

so that serious errors are attached to observations tending to the bounds of the domain, which will often lead to errors in hypothesis testing. This ratio can be visualized from figure 1. Since the tails of the assumed distribution are used to compute the probabilities of error and critical regions in all usual tests of hypothesis, it is evident that a large deviation of the assumed model from the true population distribution in this region will markedly affect the power of a test, however

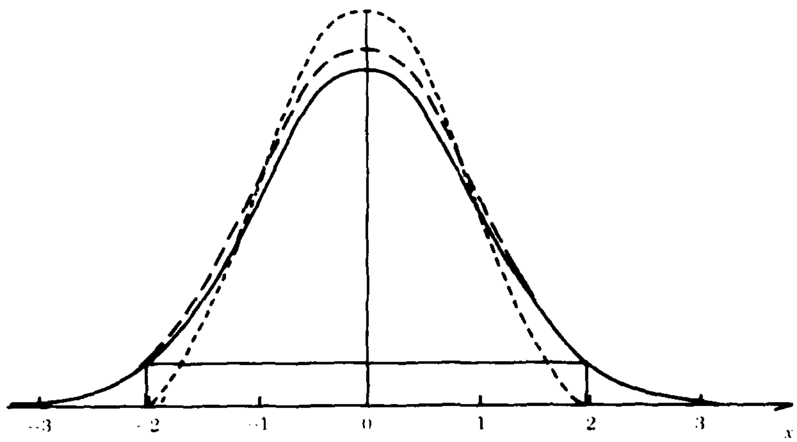

Figure 1. Superimposed density functions of normal $f(x,0,1)$ (——); truncated normal $f_c^{(t)}(x)$ for $c = 2$ (— — —); half rectified truncated normal $g_c(x)$ for $c = 2$ (- - - - -).

well behaved the approximation may be in the vicinity of the expectation. This problem was recognized under slightly different guise in 1961 by Hotelling, who observed that "Central limit theorems under various assumptions have proved convergence to normality as the number increases. For a large but fixed number the approximation of the distribution to normality is typically close within a restricted portion of its range, but bad in the tails. Yet it is the tails that are used in tests of significance."[1]

To illustrate the problem, consider the proportions, say measures p on a variable P, in samples of hake caught off the California coast in March which have anchovy larvae in their stomachs. Since the encounters between the predators and prey behave in an approximately random normal fashion,[2] the deviation about the expected proportion, say π, will be distributed approximately normal. However, the distribution of animals (presence of other potential prey but proliferation of anchovy larvae during this spawning period) implies that $p \subset [\pi - c, \pi + c]$, c a constant $<$ min $(1 - \pi, \pi)$, and obviously that the frequency of p tends to 0 as p tends to $\pi - c$ or $\pi + c$.

In such a situation, a simple maneuver will eliminate the offending property (eq. 1). Let the truncated distribution defined for $[-c, c]$ be affine-transformed so that the abscissa is translated to $f(c)$ and the ordinate "stretched" to allow for unit area under the portion of the curve above the new abscissa. Then let the curve be half-rectified (that is, drop the portion below the abscissa). The resulting curve, say $g_c(x)$, is a quasi-normal distribution with finite domain, having the property $\lim\limits_{\substack{x \to c^- \\ x \to -c^+}} g_c(x) = 0$ and yet retaining many normal properties.

Another approach to this problem would be to assume an alternative distribution with finite domain, for example, the versatile Beta or one of the time-honored Pearson clan. Some, but not extensive consideration has been given to such distributions, without auspicious success. In particular, the Beta parameters grow large as the function approximates the normal; series expansions grow complicated; and the mathematics seems to approach intractability. However, the flexibility of the Beta in dealing with the more general class of symmetric-plus-asymmetric (for example, log-normal) distributions may justify difficult mathematics and approximations; the study of the Beta and Pearson curves in this regard is encouraged in a generalization of the problem considered here.

In this report, the half-rectified truncated normal distribution is assumed, exact sampling tests of hypothesis are developed, and examples of applications are drawn from electronic component reliability and ocean temperature studies.

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[1] Superscript numbers identify references listed at end of report.

THE HALF-RECTIFIED TRUNCATION

Since any normal distribution can generally be standardized, the methodology will be developed for

$$f(x) = (2\pi)^{-1/2} \exp\left[-\frac{1}{2}x^2\right], \quad -\infty < x < \infty \tag{2}$$

We require a new function, say $g_c(c)$, or $g(x)$ for short, when c is understood. This function is approximately equal to $f(x)$ in the neighborhood of $x = 0$ but is defined only on the interval $[-c, c]$ and satisfies the boundary conditions $g(-c) = g(c) = 0$, $\lim_{x \to -c^+} g(x) = 0$, $\lim_{x \to c^-} g(x) = 0$, $c \geq 0$.

Let us define such a $g(x)$ as $g(x) = k[f(x) - f(c)]$, $-c \leq x \leq c$, where k is defined by

$$\int_{-c}^{c} g(x)\,dx = 1 \tag{3}$$

If we define $F(c) = \int_0^c f(x)\,dx$, then

$$k^{-1} = \int_{-c}^{c} [f(x) - f(c)]\,dx = 2F(c) - 2cf(c) \tag{4}$$

so that equation 2 becomes

$$g(x) = \frac{f(x) - f(c)}{2[F(c) - cf(c)]}, \quad -c \leq x \leq c \tag{5}$$

In table 1, $g(x)$ for $c = 1.2\,(0.1)\,3.0$ for $x = 0.00(0.05)\,c$ is tabulated. The dotted line in figure 1 represents $g(x)$ superimposed on the normal and truncated normal distributions for $c = 2$.

For what values of c is the half-rectified truncation appropriate? It can be seen by inspection that when c approaches 1^+, $g(x)$ deviates further and further from normal, hence losing more and more normal properties. This obviates its use, finally rendering $g(x)$ approximately a cosine or a parabola for $c \leq 1$. On the other hand, as c grows larger, $g(x) \to f(x)$. Again the use of this technique is obviated. Thus, the technique is applicable for c from somewhat greater than k, perhaps 1.2 or 1.5, to perhaps 3 or 4.

Consider the moments of this distribution. That $E_g(x) = 0$ is obvious due to the symmetry of $g(x)$ about 0. For the second moment $\sigma_g^2 = E_g(x^2)$, where, with the help of transformations, and using $I\left(\frac{c^2}{\sqrt{6}}, \frac{1}{2}\right)$ to denote the incomplete gamma distribution in Pearson's notation:[3]

$$
\begin{aligned}
E_g(x^2) = \sigma_g^2 &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} k \int_0^c x^2 \exp\left\{-\frac{1}{2}x^2\right\} dx - \frac{2kc^3}{3} f(c) \\
&= \left[\frac{1}{2} I\left(\frac{c^2}{\sqrt{6}}, \frac{1}{2}\right) - \frac{c^3}{3} f(c)\right] [F(c) - cf(c)]^{-1}
\end{aligned}
\tag{6}
$$

The relationship between σ_g^2 and $\sigma_f^2 = 1$ is not obvious. σ_g^2 is a function of c. As is requisite,

$$\lim_{c \to 0} \sigma_g^2 = 0 \tag{7}$$

and

$$\lim_{c \to \infty} \sigma_g^2 = 1 \tag{8}$$

which are easily shown by L'Hospital's rule. For finite $c > 0$, let us return to the form of equation 6, which may be written as

$$\sigma_g^2 = 1 - \frac{\frac{c^3}{3} f(c)}{\int_0^c x^2 \exp\left\{-\frac{1}{2}x^2\right\} dx} \tag{9}$$

But when $c > 0$, both denominator and numerator of the fraction in equation 9 are positive; hence $\sigma_g^2 < \sigma_f^2 = 1$ for finite $c > 0$.

Table 2 gives values of σ_g^2 for $C = 0.80\ (0.05)\ 4.00$. Computations were performed on the University of Connecticut Computer Center IBM 7040. Since $\sigma_f^2 = 1$, each table entry will also represent $\frac{\sigma_g^2}{\sigma_f^2}$.

A TEST OF MEANS, KNOWN VARIANCE

Let us consider the use of $g(x)$ for a test of means where $\sigma_{\dot{x}}^2$ is known. Let $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ be a sample of n independent observations on $\dot{x}$, where $\dot{x} \stackrel{\sim}{-}$† $n(\mu, \sigma)$, but $\dot{x} \in [\mu - c\sigma, \mu + c\sigma]$. Let $\bar{\dot{x}} = \frac{1}{n}\sum_{1}^{n} \dot{x}_i$. It follows from $-nc\sigma \leq \sum_{1}^{n} (\dot{x}_i - \mu) \leq nc\sigma$, and naming $x = (\bar{\dot{x}} - \mu)/\sigma$, that $-c \leq x \leq c$, where x is now a random variable for which the transformation from (2) to (3) is appropriate. Thus, the theory here will serve for a test on $\bar{\dot{x}}$.

In this case,

$$P\left[\frac{\bar{\dot{x}} - \mu}{\sigma} \leq \xi\right] = P[x \leq \xi] = \int_{-c}^{\xi} g(x)\,dx = \frac{F(\xi) - F(-c)}{2F(c)} = \frac{(\xi + c) f(c)}{2 c f(c)} \qquad (10)$$

Let us define $G(\xi) = \int_0^{\xi} g(x)\,dx$. In table 3 $G(x)$ for $c = 1.2\ (0.1)\ 3.0$ for $x = 0.00\ (0.05)\ c$ is tabulated. Computations were performed on a PTP-APS-augmented Mathatronics 8-48S. Then from equation 10,

$$P[x \leq \xi] = \frac{1}{2} + G(\xi),\ \xi \geq 0 \qquad (11)$$

$$= \frac{1}{2} - G(-\xi),\ \xi < 0$$

In some cases, more general use arises from the Central Limit Theorem. For any $\dot{x}$ $IID(\mu, \sigma)$, $\dot{x} \in [\mu - c^*\sigma, \mu + c^*\sigma]$, $-c = -\sqrt{n}\,c^* \leq \frac{\bar{\dot{x}} - \mu}{\sigma/\sqrt{n}} \leq \sqrt{n}\,c^* = c$, where $\frac{\bar{\dot{x}} - \mu}{\sigma/\sqrt{n}}$ will tend to normality. For small n, there is a region of $c = \sqrt{n}\,c^*$ in which the C.L.T. has begun to cause normality tendencies but where n has not adequately reduced the variance (relative to c^*), where this theory would again be helpful. The nature of this region has not been investigated. However, a comparison of table 3 with a table of $\int_0^x n(t; 0, 1)\,dt$ shows that for small c, certainly $c < 3$, the incomplete distributions differ sufficiently to imply the desirability of using this method.

It may be of interest to compare Type I Error probabilities between the two tests. Two approaches to comparison are apparent: the change in critical value could be studied for fixed, or the change in α could be studied for a fixed critical value. The first would affect the nature of the decision resulting from

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† The sign $\stackrel{\sim}{-}$ signifies "is distributed approximately."

changing tests while holding a constant error risk; the second would affect the risk of error resulting from changing tests while retaining the critical value. The first is used later in considering examples. The second would not occur in practice, but would yield valuable insight into the relationship between the two types of test; therefore it is considered here.

Suppose we use a normal distribution test based on some Type I Error probability, say α_N; we find a certain critical value, say v_c, which separates the rejection region from the acceptance region. However, suppose the conditions set forth in this report are exactly satisfied. Then α_N is not the true Type I Error probability; a correct α_H is associated with the half-rectified truncated normal distribution, or "*H*-test." Further, $\alpha_N > \alpha_H$, so that a normal test used when the *H*-test is appropriate will yield a Kimball[4] Type III Error†, κ, representing the difference, or $\alpha_N - \alpha_H = \kappa$.

Consider $c = 2.8$, as occurred in one application. To find α_H, let us enter table 3 under $c = 2.8$ and $x = v_c$. The tabulated (or interpolated) value is the area under the probability curve from 0 to v_c, so that 1-entry is the one-tailed error probability. Then $\alpha_H = 2$ (1-entry). Appearing below are α_N, v_c, α_H, and κ ($= \alpha_N - \alpha_H$):

α_N	0.2000	0.1000	0.0500	0.0200	0.0100	0.0050
v_c	1.28	1.64	1.96	2.33	2.57	2.81
α_H	0.1806	0.0820	0.0334	0.0078	0.0014	0.0000
κ	0.0194	0.0180	0.0166	0.0122	0.0086	0.0050
κ/α_H	0.1074	0.2195	0.4970	1.5641	6.1429	∞

While a κ is present in each case, it decreases with α and may not appear crucial at first glance. More important for an interpretation of κ is its relation to α, the error probability usually considered. Thus, the last row gives the ratio κ/α_H, which can be seen to increase seriously as α decreases. In any event, κ is large enough so that serious doubts are cast on the advisability of using a normal test when the probability distribution diminishes to zero at the ends of a bounded sample range.

† Kimball defines the Type III Error as "the error committed by giving the right answer to the wrong problem." An equivalent definition which is more susceptible to measurement might be "the error committed by making an erroneous decision due to erroneous assumptions." Let us name the probability of this error κ in honor of Kimball, using the Greek letter for harmony with α and β, the Type I and II Error probabilities.

A TEST OF MEANS, UNKNOWN VARIANCE

Thus far we have assumed a normal distribution with (possibly) known mean μ and known variance σ^2. Let us now consider the case in which σ^2 is unknown and must be estimated.

Suppose we have a sample of n independent observations $\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n$ arising from the distribution of $\dot{x}$ distributed as normal with zero mean and unit standard deviation, denoted $\sim n(\mu, \sigma)$, and σ is unknown. Let

$$s^2 = \frac{1}{n} \sum_{1}^{n} (\dot{x}_i - \mu)^2 \tag{12a}$$

or

$$s^2 = \frac{1}{n-1} \sum_{1}^{n} (\dot{x}_i - \bar{\dot{x}})^2 \tag{12b}$$

depending on μ either known or estimated by $\bar{\dot{x}} = \frac{1}{n} \sum_{1}^{n} \dot{x}_i$, and

$$x = \frac{\bar{\dot{x}} - \mu}{\sigma} \tag{13}$$

As in equation 2, x will be distributed as $n(0, 1)$. But now we shall impose the restriction $-c \leq x \leq c$ on the possibility space. Then as before $g(x)$ is distributed as in equation 5. Now let

$$y_i = \frac{(\dot{x}_i - \mu)^2}{\sigma^2} \tag{14a}$$

or

$$y_i = \frac{(\dot{x}_i - \bar{\dot{x}})^2}{\sigma^2} \tag{14b}$$

depending on μ known or unknown, respectively. We desire the probability function, say $h(t)$, (which now defines $h(t)$), of

$$t = \sqrt{\frac{\theta}{n}} \frac{x}{\sqrt{v}} \tag{15}$$

where

$$v = \sum_{1}^{n} y_i = \frac{\theta s^2}{\sigma^2} \tag{16}$$

where θ is degrees of freedom, namely, n or $n-1$, depending on μ known or unknown.

Now if the distribution function for equation 15, $H(t_0; \theta, c) = \int_0^{t_0} h(t; \theta, c)\, dt$, approaches the distribution function for Student's t rapidly as n increases, Student's t will serve as an adequate approximation, and the derivation of H will be unnecessary. Certainly H approaches the distribution function of Student's t for n as c increases. However, for c in practical ranges of, say, 1.2 to 3.0, a comparison of the to-be-derived H tabulated in table 4 with a Student's t-table will show large differences. For example, if $\theta = 10$ and $c = 2.0$, the value of t_0, for which $0.5 - H(t_0) = 0.05 = \alpha$ is 1.682. If Student's t were used erroneously in this situation for a nominal $\alpha = 0.05$, t_0 would be 1.812, and the true α for a t_0 of 1.682 would be approximately 0.08, an error of 60 percent.

Let us, then, derive h and tabulate H. Transforming $g(x)$ of equation 5 by the transformation (eq. 14a, b), we may easily show

$$g_1(y_i)\,dy_i = \frac{2^{-3/2}\pi^{-1/2}}{F(c) - cf(c)}\, y_i^{-1/2}\left(e^{-y_i/2} - e^{-c^2/2}\right) dy_i,\ 0 \le y_i \le c^2 \tag{17}$$

Since y_i is independent of y_j, $i, j = 1, 2, \ldots, n$, $i \neq j$,

$$\begin{aligned} g_2(y_1, \ldots, y_n) = {} & \frac{2^{-3n/2}\pi^{-n/2}}{[F(c) - cf(c)]^n} \prod_{1}^{n} y_i^{-1/2} \Bigg[\prod_{1}^{n} e^{-y_i/2} - e^{-c^2/2} \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i \neq j}}^{n} e^{-y_i/2} \\ & + e^{-2c^2/2} \sum_{j=1}^{n-1} \sum_{\substack{\ell=2 \\ \ell > j}}^{n} \prod_{\substack{i=1 \\ i \neq j \\ i \neq \ell}}^{n} e^{-y_i/2} - \ldots + (-1)^n e^{-nc^2/2} \Bigg] \end{aligned} \tag{18}$$

Let us make the transformation (eq. 16), such that $y_1 = v - \sum_{2}^{n} y_i$, $dy_1 = dv$, and integrate out $y_2, \ldots, y_n$, whence

$$g_3(v) = \frac{2^{-3n/2}\pi^{-n/2}}{[F(c)-cf(c)]^n} e^{-v/2} \int_0^{c^2} \cdots \int_0^{c^2} \left(v - \sum_2^n y_i\right)^{-1/2} \prod_2^n y_i^{-1/2} \Bigg[1 \tag{19}$$

$$- e^{-c^2/2}\left(\sum_2^n e^{y_i/2} + e^{\frac{1}{2}\left(v - \sum_2^n y_i\right)}\right) + e^{-2c^2/2}\left(\sum_{j=2}^{n-1} \sum_{i=3}^{n} e^{\frac{1}{2}(y_i + y_j)}\right.$$

$$\left. + \sum_2^n e^{-y_i/2} e^{\frac{1}{2}\left(v - \sum_2^n y_i\right)}\right) - \cdots + (-1)^n e^{v/2} e^{-nc^2/2}\Bigg] \prod_2^n dy_i$$

We note that each integral of the expansion (eq. 19) is nearly in the form of a generalized Dirichlet integral. Let $y_i = vu_i$, $u_i \geq 0$, which satisfies one condition for the Dirichlet integral. Also $\sum y_i = v \sum u_i = v$, so that $\sum u_i = 1$, which satisfies another condition. Since $u_i = \frac{y_i}{v} = \frac{y_i}{y_i + \sum_{i \neq j} y_i}$, it can be seen that $u_i \leq 1$, subject to the restriction that $\sum u_i = 1$. The possibility space of equation 19 becomes an $(n-1)$-dimensional hypercube from zero to the intersection with the hyperplane $\sum u_i = 1$.

$$g_3(v) = \frac{2^{-3n/2}\pi^{-n/2}}{[F(c)-cf(c)]^n} v^{\frac{n-2}{2}} e^{-\frac{v}{2}} \underset{\sum u_i \leq 1}{\int_0^1 \cdots \int_0^1} \left(1 - \sum_2^n u_i\right)^{-1/2} \prod_2^n u_i^{-1/2} \Bigg[1$$

$$- e^{-c^2/2}\left(\sum_2^n e^{vu_i/2} + e^{\frac{v}{2}\left(1 - \sum_2^n u_i\right)}\right) + e^{-2c^2/2}\left(\sum_{j=2}^{n-1} \sum_{\substack{i=3 \\ i \neq j}}^{n} e^{\frac{v}{2}(u_i + u_j)}\right. \tag{20}$$

$$\left. + \sum_2^n e^{-vu_i/2} e^{\frac{v}{2}\left(1 - \sum_2^n u_i\right)}\right) - \cdots + (-1)^n e^{v/2} e^{-nc^2/2}\Bigg] \prod_2^n du_i$$

If the integral in equation 20 is expanded into a sum of integrals, the first is clearly the generalized Dirichlet integral:

$$\underset{\sum u_i \le 1}{\int_0^1 \cdots \int_0^1} \left(1-\sum_2^n u_i\right)^{-1/2} \prod_2^n u_i^{-1/2} \prod_2^n du_i \quad \frac{\Gamma^n(1/2)}{\Gamma(n/2)}$$

a constant. The remaining integral may be written

$$\sum_{\ell=1}^{n} (-1)^\ell e^{-\ell c^2/2} \sum_{\substack{j_\ell=1\\ j_\ell<j_{\ell-1}}}^{n-\ell+1} \cdots \sum_{\substack{j_2=\ell-1\\ j_2<j_1}}^{n-1} \sum_{j_1=\ell}^{n} \int_0^1 \qquad (21)$$

$$\cdots \int_0^1 e^{\frac{v}{2}(u_{j_1}+u_{j_2}+\cdots+u_{j_{\ell-1}}+u_{j_\ell})} \prod_{i=2}^{n} u_i^{-1/2} \left(1-\sum_2^n u_i\right)^{-1/2} \prod_2^n du_i$$

where $\left(1-\sum_2^n u_i\right)$ is represented in the exponential term by u_1 for notational compactness. If we expand the exponent in expression 21 such that

$$e^{\frac{v}{2}(u_{j_1}+u_{j_2}+\cdots+u_{j_{\ell-1}}+u_{j_\ell})} = 1 + \frac{v}{2}(u_{j_1}+\cdots+u_{j_\ell}) + \frac{v^2}{2^2 2!}(u_{j_1}+\cdots+u_{j_\ell})^2 + \cdots$$

then expression 21 clearly becomes a sum of Dirichlet integrals and (adding $\Gamma^n(1/2)/\Gamma(n/2)$ to form $g_3(v)$) may be written

$$g_3(v) = \frac{\Gamma^n(1/2)}{\Gamma(n/2)} + \sum_{\ell=1}^{n} (-1)^\ell e^{-\ell c^2/2} \sum_{\substack{j_\ell=1\\ j_\ell<j_{\ell-1}}}^{n-\ell+1} \cdots \sum_{\substack{j_2=\ell-1\\ j_2<j_1}}^{n-1} \sum_{j_1=\ell}^{n} \sum_{r=0}^{\infty} \underset{\sum_{\alpha=1}^{\ell-1} r_\alpha \le r}{\sum_{r_1=0}^{r} \sum_{r_2=0}^{r}} \qquad (22)$$

$$\cdots \sum_{r_{\ell-1}=0}^{r} \frac{v^r}{2^r r_1!\, r_2! \cdots \left(r-r_1-\cdots-r_{\ell-1}\right)!} \; \frac{\pi^{\frac{n-\ell}{2}}\, \Gamma(r_1+1/2) \cdots \Gamma\left(1/2+r-\sum_{\alpha=1}^{\ell-1} r_\alpha\right)}{\Gamma\left(\frac{n}{2}+r\right)}$$

which may easily be shown to be a convergent infinite series.

To find the joint function of x, c, it would be convenient to have x, c independent. Although they are not, since the normal is the only distribution having this property (see, for example, Laha[5]), $g(x) \to$ normality as c grows large, which suggests to the intuition that x, c may be "almost" independent, i.e., the marginal distribution of one of them may approximate its distribution conditional upon the other.

To this end, a Monte Carlo study was undertaken. For each of various n and c, 400 samples of x_i of size n were randomly generated from $g(x)$ by the University of Connecticut's IBM *System 360 Model 65 (supported in part by NSF* Grant GP-1819). x and c were computed and their frequencies tabulated in steps of $0.1n$ for both. The sums of rows and columns gave Monte Carlo approximations to marginal distributions of x and c. χ^2 contingency tests were carried out with $\alpha = 0.05$. The null hypothesis of independence was rejected for (n, c) as small as $(10, 1.5)$ but was accepted for either larger n or larger c, implying that if both n, c become small, significant dependence obtains, but the dependence is not significant otherwise.

As further evidence, correlation was considered. The contingency tables indicated that no correlation other than linear was present. Values of x, c were drawn at random and a computer (PTP-APS-augmented Mathatronics 8-48S) simulation of linear correlation of x, c effected for various c. For $c = 1.2$, $r^2 = 0.017$ ($r \doteq 0.13$). r^2 decreased as c increased until, for $c = 2.0$, $r^2 \doteq 0.004$ ($r \doteq 0.06$), which is consistent with and reinforces the conclusion that independence is a safe assumption for $c > 1.5$ and for any c if n is other than quite small.

Assuming the independence of x, c, let us obtain the joint function of x, c. From equations 5 and 22

$$h_1(x, c) = \frac{f(x) - f(c)}{2[F(c) - cf(c)]} g_3(c) \tag{23}$$

We desire the probability function of t, where t is as in equation 15. We know $x = (n/\theta)^{1/2} t v^{-1/2}$, and

$$h_1(t,v) = \frac{f\left[(n/\theta)^{1/2} t v^{-1/2}\right] - f(c)}{2[F(c) - cf(c)]} \left(\frac{n}{\theta}\right)^{1/2} v^{-1/2} g_1(v)$$

Expanding and integrating out v, we find

$$\begin{aligned}
h(t) = \frac{(n/\theta)^{1/2}}{2\sqrt{2\pi}[F(c) - cf(c)]} \Bigg\{ & \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^{nc^2} v^{1/2} e^{-\frac{n}{2\theta} t^2} dv \qquad (25) \\
& + \frac{\pi^{n/2} e^{-\frac{1}{2}c^2}}{\Gamma(n/2)} \int_0^{nc^2} v^{1/2}\, dv + \int_0^{nc^2} v^{1/2} e^{-\frac{n}{2\theta} t^2} \sum_{\ell=1}^{n} (-1)^\ell e^{-\ell c^2/2} \\
& \cdot \sum_{\substack{j_\ell = 1 \\ j_\ell < j_{\ell-1}}}^{n-\ell+1} \cdots \sum_{\substack{j_2 = \ell-1 \\ j_2 < j_1}}^{n-1} \sum_{j_1 = \ell}^{n} \sum_{r=0}^{\infty} \sum_{r_1=0}^{r} \cdots \sum_{\substack{r_{\ell-1}=0 \\ \sum_{\alpha=1}^{\ell-1} r_\alpha \le r}}^{r} \frac{v^r}{2^r r_1! \cdots \left(r - \sum_{\alpha=1}^{\ell-1} r_\alpha\right)!} \\
& \cdot \frac{\pi^{\frac{n-\ell}{2}} \Gamma\left(r_1 + \frac{1}{2}\right) \cdots \Gamma\left(\frac{1}{2} + r - \sum_{\alpha=1}^{\ell-1} r_\alpha\right)}{\Gamma\left(\frac{n}{2} + r\right)} dv \\
& + e^{-\frac{1}{2}c^2} \int_0^{nc^2} v^{1/2} \sum_{\ell=1}^{n} (-1)^\ell e^{-\ell c^2/2} \sum_{\substack{j_\ell = 1 \\ j_\ell < j_{\ell-1}}}^{n-\ell-1} \cdots \sum_{j_1 = \ell}^{n} \sum_{r=0}^{\infty} \sum_{r_1=0}^{r} \cdots \sum_{\substack{r_{\ell-1}=0 \\ \sum_{\alpha=1}^{\ell-1} r_\alpha \le r}}^{r} \\
& \cdot \frac{v^r}{2^r r_1! \cdots \left(r - \sum_{\alpha=1}^{\ell-1} r_\alpha\right)!} \cdot \frac{\pi^{\frac{n-\ell}{2}} \Gamma\left(\frac{1}{2} + r_1\right) \cdots \Gamma\left(\frac{1}{2} + r - \sum_{\alpha=1}^{\ell-1} r_\alpha\right)}{\Gamma\left(\frac{n}{2} + r\right)} dv \Bigg\}
\end{aligned}$$

In equation 25, let us make the transformation $u = \frac{n}{2\theta} v t^2$ in the first and third integrals. Then

$$h(t) = \frac{(n/\theta)^{1/2}}{2\sqrt{2\pi}\,[F(c) - cf(c)]} \left\{ \frac{\pi^{n/2}}{\Gamma(n/2)} \left(\frac{2\theta}{n}\right)^{3/2} t^{-3} \int_0^{\frac{n^2c^2}{2\theta}t^2} u^{1/2} e^{-u}\, du \right. \tag{26}$$

$$+ \frac{\pi^{n/2} e^{-c^2/2}}{\Gamma(n/2)} \frac{2}{3} n^{3/2} c^3 + \sum_{\ell=1}^{n} (-1)^\ell e^{-\ell c^2/2} \sum_{\substack{j_\ell=1 \\ j_\ell > j_{\ell-1}}}^{n-\ell+1} \cdots \sum_{j_1=\ell}^{n} \sum_{\substack{r=0 \\ \sum_{\alpha=1}^{\ell-1} r_\alpha \le r}}^{\infty} \cdots \sum_{r_{\ell-1}=0}^{r}$$

$$\frac{\pi^{\frac{n-\ell}{2}} \Gamma\left(\frac{1}{2} + r_1\right) \cdots \Gamma\left(\frac{1}{2} + r - \sum_{\alpha=1}^{\ell-1} r_\alpha\right)}{\Gamma\left(\frac{n}{2} + r\right) 2^r r_1! \cdots \left(r - \sum_{\alpha=1}^{\ell-1} r_\alpha\right)!} \left(\frac{2\theta}{n}\right)^{r+3/2} t^{-2r-3} \int_0^{\frac{n^2c^2}{2\theta}t^2} u^{r+1/2} e^{-u}\, du$$

$$+ e^{1/2 c^2} \sum_{\ell=1}^{n} (-1)^\ell e^{-\ell c^2/2} \sum_{\substack{j_\ell=1 \\ j_\ell > j_{\ell-1}}}^{n-\ell+1} \cdots \sum_{j_1=\ell}^{n}$$

$$\left. \sum_{\substack{r=0 \\ \sum_{\alpha=1}^{\ell-1} r_\alpha \le r}}^{\infty} \cdots \sum_{r_{\ell-1}=0}^{r} \frac{\pi^{\frac{n-\ell}{2}} \Gamma\left(\frac{1}{2} + r_1\right) \cdots \Gamma\left(\frac{1}{2} + r - \sum_{\alpha=1}^{\ell-1} r_\alpha\right)}{\Gamma\left(\frac{n}{2} + r\right) 2^r r_1! \cdots \left(r - \sum_{\alpha=1}^{\ell-1} r_\alpha\right)!} \frac{n^{r+3/2} c^{2r+3}}{r + 3/2} \right\}$$

which may be simplified to

$$h(t) = \frac{(n/\theta)^{1/2}}{2\sqrt{2\pi}[F(c)-cf(c)]} \left\{ \frac{\pi^{n/2}}{\Gamma(n/2)} \left[(\theta/n)^{3/2} t^{-3} I\left(\frac{n^2c^2}{\sqrt{6}\,\theta} t^2, \frac{1}{2}\right) \right.\right. \tag{27}$$

$$\left. - \frac{2}{3} n^{3/2} c^3 e^{-c^2/2} \right] + \sum_{\ell=1}^{n} (-1)^{\ell} e^{-\ell c^2/2} \sum_{\substack{j_\ell = 1 \\ j_\ell < j_{\ell-1}}}^{n-\ell+1}$$

$$\cdots \sum_{\substack{j_2 = \ell-1 \\ j_2 < j_1}}^{n-1} \sum_{j_1=\ell}^{n} \sum_{r=0}^{\infty} \sum_{\substack{r_1=0 \\ \sum_{\alpha=1}^{\ell-1} r_\alpha \le r}}^{r} \sum_{r_2=0}^{r} \cdots \sum_{r_{\ell-1}=0}^{r}$$

$$\cdot \frac{\pi^{\frac{n-\ell}{2}} \Gamma\left(\frac{1}{2}+r_1\right) \cdots \Gamma\left(\frac{1}{2}+r-\sum_{\alpha=1}^{\ell-1} r_\alpha\right)}{\Gamma\left(\frac{n}{2}+r\right) 2^r r_1! \cdots \left(r-\sum_{\alpha=1}^{\ell-1} r_\alpha\right)!}$$

$$\left.\left[\Gamma\left(r+\frac{3}{2}\right)\left(\frac{2\theta}{n}\right)^{r+3/2} t^{-2r-3} I\left(\frac{n^2c^2}{\sqrt{6}\,\theta} t^2, r+\frac{1}{2}\right) - \frac{e^{-c^2/2} n^{r+1/2} c^{2r+1}}{r+3/2}\right]\right\}$$

In order to perceive the functional form of t unobscured by the several constants, equation 27 could be written, with $k_1, \ldots, k_4$ as various constants, as

$$h(t) = k_1 \left[k_2 + k_3 t^{-3} I\left(\frac{n^2c^2}{\sqrt{6}\,\theta} t^2, 1/2\right) + \{\text{sum of terms of } \{ k_4 t^{-2r-3} I\left(\frac{n^2c^2}{\sqrt{6}\,\theta} t^2, r+1/2\right) \} \} \right]. \tag{28}$$

Desired for a test of significance on the statistic t will be a critical region of size α and a value of t, say t_0, such that

$$\int_{t_0}^{\infty} h(t) = \alpha \text{ for a right-tailed test} \tag{29}$$

$$= \alpha/2 \text{ for a two-tailed test.}$$

Let us define $H(t_0; \theta, c) = \int_0^{t_0} h(t; \theta, c)\, dt$. Table 4 gives $H(t; \theta, c)$ for $\theta = 1\,(1)\,30$ for $c = 1.2\,(0.1)\,3.0$ for $\alpha = 0.10, 0.05, 0.02$, and 0.01. Computations were performed on an APS-PTP-augmented Mathatronics 8-48S. From table 4, values of the sort in equation 29 desired for significance testing may be found.

TESTS OF VARIANCE

If it should be desired to test an estimated variance s^2 against a known variance σ^2, the statistic is v as given in equation 16, and the acceptance or rejection of the hypothesis H_0: $s^2 = \sigma^2$, σ^2 known, will be based on the critical region of v obtained from an integration of $g_1(v)$ given in equation 22, a convergent infinite series.

If it should be desired to test two estimated variances of two independent samples, that is H_0: $s_1^2 = s_2^2 = \sigma^2$, σ^2 unknown, then the ratio $F^* = v_1/v_2$, $v_i = \dfrac{\theta_i s_i^2}{\sigma^2}$, $i = 1, 2$, θ_i d.f. associated with sample i, may be used and the probability distribution of F^* may be obtained from using $g_1(v)$ twice and the transformation $F^* = v_1/v_2$.

The forms of the above distributions, however, are complicated in nature. We may feel that they are too complicated to make tabulation worthwhile. If a use of enough importance arises, the appropriate distribution may be adequately approximated much more easily by specifying the known parameters of the given case.

AN APPLICATION IN ELECTRONIC COMPONENT RELIABILITY EVALUATION

Most systems are characterized by the property that a malfunction of a component causes a degradation or malfunction of the entire system. Most reasonable measures or sets of measures used to evaluate the quality and/or effectiveness of the system[6] necessarily include a consideration of the probability function associated with the time out of operation of the system. The functional form of this system outage time distribution must be inferred from experimental data. There have been different functions inferred variously in the literature; the disparity is due largely to the differing natures of different systems. The differences may be analyzed by representing the system by a model of the form $t = \sum_{i=1}^{5} x_i$, where

t is length of time system is out of operation;

x_1 is length of time of search for location of trouble;

x_2 is administrative time, including waiting time for permission or assignment to repair, waiting time for funds to cover repair, and any undefined waste time;

x_3 is logistic time, such as waiting time for a necessary part, waiting time for the arrival of a repairman, etc.;

x_4 is test and check-out of the new or repaired component;

x_5 is active repair time of the malfunctioned component

<u>or</u>

pull-replace time of the malfunctioned component.

This report is primarily concerned with x_5, hereafter designated x. It is possible for x to be quite small in the event that an outage is predicted, that is, when there is advance warning that a component is about to malfunction. This small x is more the product of chance than plan since the operator of the system cannot, in general, predict the malfunction of a component. It has been observed that such prediction occurs infrequently. Also a very large x, that is, an extensive active repair time, has been observed to occur very infrequently. Such properties, plus the obviously desirable property that the repair rate (the first derivative of the function) is continuous, implies an approximately bell-shaped function. This function is close enough to symmetric that it has usually been adequately approximated by the normal curve. Although the probability function of x is occasionally right-skewed, which necessitates another functional form, such as log normal or gamma, it is the more common symmetric bell-shaped curve that is of interest here.

Consider the pull-replace times for malfunctioned components of a Naval electronics system. In the case considered, eight tests were performed on each of two competitive system configurations. It was observed that the scatter of experimental times was approximately symmetric about the mean of each and that the sample variance was very close on each.

The traditional method of testing the significance of the difference between the disparate means would have been to assume that the pull-replace times for systems A and B respectively arose randomly and independently from normal probability functions having the same variance, and then to test the null hypothesis H_0: the mean of the normal distribution of system A is not different from the mean of the normal distribution of system B. The appropriate test would have been thought to be the two-tailed Student's t- test. Student's t at the 0.05 level of significance has the value 2.15 for 14 degrees of freedom. The pooled sample variance s^2 using

14 degrees of freedom was found to be 5.3094. The two sample means were found to be $\bar{x}_A = 7.96$ minutes and $\bar{x}_B = 5.72$ minutes. The computation of t (for $n_A = n_B = 8$) shows that

$$t = \frac{\bar{x}_A - \bar{x}_B}{s\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}} = \frac{2.24}{2.30/2} = 1.95 \tag{30}$$

which is less than 2.15, from which it would have been concluded that the two means were not different.

From an investigation into the nature of the pull-replace action the engineers and scientists involved agreed that the pull-replace time could not possibly deviate from the average by more than five minutes. It was assumed that the variances were the same for A and B, and the observation that the sample variances were similar provided some justification for this assumption. The assumptions behind using the transformed normal distribution of this report seem rather well justified. Let us consider an equivalent t-test, using, however, table 4 rather than the Student table.

Since one-half the range† is $cs = 5$, $c = 2.223$, which is close enough to use the tabulation for $c = 2.3$. According to table 4, $H(t_0$; 14 d.f., 2.3) for $\alpha = 0.05$ yields $t_0 = 1.79$. The fact that $t = 1.95 > 1.79$ implies that the difference between the pull-replace times for the two systems is statistically significant and that system B is better, an implication which was not present under the erroneous assumption of infinite [illegible] of the variable.

AN APPLICATION IN OCEAN DATA ANALYSIS

Since 1957, the Canadian government has maintained a weather ship continuously (except for short replacement intervals) near a fixed North Pacific location (lat 50° N, long 145° W), known as Station "P" or PAPA. On many days the ship has observed seawater temperature at various approximately standard depths, by means of Nansen casts; each time the location of the ship was recorded to accuracy of minutes.

† Recall that the sample range is defined as lying from $-c$ standard deviations to $+c$ standard deviations.

The position was read periodically and the ship altered course to head to the 50° 00′ N, 145° 00′ W location whenever it was found to be off. An investigation of the positioning strategy, including the speeds and course patterns involved in repositioning, current effects, etc., dictated that conditions *could not* combine to force the ship more than 15′ from its mean location without erroneous navigation or storm conditions, and that the probability of deviant positioning approached 0 continuously to the 15′ deviation as a limit. Since Station P is in a position of unstable temperature profile (functional relationship between temperature and depth) type, a small location deviation may imply serious temperature measurement errors. A question preceding analyses on the temperature profiles is: Does the ship's mean location at recording change with time? A part of the testing to answer this question is quote 1.

The position 50° N, 145° W was considered as (0, 0) with deviations being recorded in minutes. (Thus, 49°58′ N, 145° 11′ W became the couple (-2,11).) Latitude and longitude were considered separately. The last data year was 1963. Mean latitude for 1963 was compared with that of 1962, no significant difference was found, and the data were pooled. Mean latitude for 1962-3 was 0.23 with n_1 81 and for 1961 was -0.41 with n_2 73. The t-statistic was computed to be 0.86, not significant under either normal probabilities or the proposed $G(t)$. Similarly, longitude was pooled for 1962 and 1963, having a mean of -1.11; for 1961, it was 0.38. The pooled standard deviation was 6.44 and the t statistic 1.43. From normal probability tables, a two-tailed test of the null hypothesis: $\mu_{1962\text{-}3}$ μ_{1961} for α 0.10 yields a critical value of 1.645, implying acceptance of the hypothesis. However, consider a similar test based on the G-table. Here c_s 6.44c 15, so $\hat{c}$ 2.33. $G_{2,3}(x)$ yields x 1.43 by linear interpolation, so that, interestingly, the statistic falls just on the critical value. A recomputation carrying more significant digits implies an acceptance of the hypothesis, but the results give rise to suspicion. An investigation of the data showed two outliers, apparently traceable to storm effects. Further, the temperature profiles at those locations differed from those expected for the respective dates; it was well they were discovered. A recomputation of the 1961 longitudes with the outliers removed, n_2 now 71, yielded a mean of -0.56, a new pooled standard deviation of 5.14, and a new t of 0.54. The conclusion was reached that the remaining 1961 data arose from the same location population as the 1962-3 data and that location deviation is a crucial factor in the analysis of Station P temperature data.

CONCLUSIONS

Two traditional approaches are in use for hypothesis testing of data samples drawn from populations with approximately normal distributions but with finite domains: (a) assumption of a distribution that closely approximates the sample, such as a Beta distribution or one of the Pearson curves, and (b) truncation (removing the tails) of an assumed normal population. Approach (a) leads to intractable mathematics, and it is difficult to justify the appropriate curve and parameters. Approach (b) leads to serious errors in hypothesis testing if the frequency curve approaches zero at the bounds of the domain, since the tails of the assumed distribution are used in computing probabilities of error and in separating decision regions.

In this study, a transformation is performed on an assumed truncated normal distribution so that many of the desirable statistical properties of normal distribution are retained, yet the data sample distribution in the tails is more closely matched than is possible through use of the traditional approaches.

The major results of the study are:

1. *Development of sampling* theory and tests of hypothesis for:

    a. mean tests on samples with known variance,

    b. mean tests on samples with estimated variance,

    c. *tests of variance;*

2. application of the technique to two data samples, one concerned with electronic component reliability evaluation, the other with ocean data analysis;

3. preparation of extensive tables to make possible practical use of the method.

RECOMMENDATIONS

In application of statistical tests to specific problems, the satisfaction of assumptions should be examined when normal tests, such as t-tests or F-tests, are used. When assumptions are violated by domain restrictions, the method described in this study may be preferable.

Sometimes the more realistic Beta or Pearson distribution may be used, but results obtained from their use should be compared with those obtained by means of the approach that this report offers.

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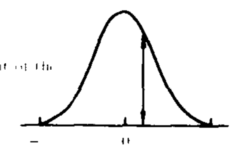
TABLE 1. PROBABILITY DENSITIES FOR
THE $g(x)$-FUNCTION (Continued through page 28)



Tabulated is the height of the probability function.

x	1.2		1.3		1.4		1.5		1.6	
	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05
0.0	0.6740	0.6723	0.6306	0.6292	0.5944	0.5923	0.5637	0.5627	0.5378	0.5369
0.1	.6677	.6595	.6253	.6184	.5899	.5839	.5597	.5545	.5342	.5296
0.2	.6480	.6338	.6087	.5968	.5756	.5653	.5472	.5382	.5230	.5150
0.3	.6164	.5959	.5821	.5649	.5527	.5379	.5271	.5141	.5051	.4935
0.4	.5732	.5475	.5458	.5242	.5214	.5028	.4997	.4834	.4807	.4661
0.5	.5199	.4896	.5003	.4754	.4828	.4608	.4658	.4465	.4504	.4332
0.6	.4577	.4241	.4485	.4202	.4377	.4134	.4262	.4049	.4151	.3961
0.7	.3888	.3520	.3906	.3595	.3878	.3611	.3825	.3591	.3761	.3552
0.8	.3144	.2759	.3279	.2955	.3339	.3060	.3352	.3107	.3339	.3120
0.9	.2367	.1972	.2625	.2292	.2776	.2490	.2858	.2607	.2898	.2674
1.0	.1574	.1175	.1957	.1622	.2202	.1913	.2354	.2101	.2448	.2222
1.1	.0780	.0385	.1289	.0956	.1627	.1341	.1850	.1599	.1998	.1774
1.2	.0000	.0000	.0632	.0313	.1061	.0787	.1354	.1113	.1555	.1341
1.3			.0000	.0000	.0518	.0255	.0877	.0647	.1130	.0924
1.4					.0000	.0000	.0423	.0021	.0725	.0532
1.5							.0000	.0000	.0347	.0017
1.6									.0000	.0000
1.7										
1.8										
1.9										
2.0										
2.1										
2.2										
2.3										
2.4										
2.5										
2.6										
2.7										
2.8										

TABLE 1. PROBABILITY DENSITIES FOR THE $g_c(\lambda)$-FUNCTION

λ	1.7		1.8		1.9		2.0		2.1	
	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05
0.0	0.5159	0.5150	0.4969	0.4961	0.4807	0.4800	0.4670	0.4663	0.4554	0.4547
0.1	.5126	.5084	.4939	.4901	.4780	.4744	.4644	.4610	.4529	.4497
0.2	.5025	.4952	.4846	.4779	.4693	.4631	.4563	.4504	.4452	.4397
0.3	.4862	.4757	.4697	.4601	.4555	.4465	.4433	.4349	.4329	.4249
0.4	.4641	.4509	.4494	.4372	.4366	.4253	.4255	.4150	.4161	.4061
0.5	.4367	.4211	.4242	.4099	.4132	.4000	.4036	.3911	.3953	[illegible]
0.6	.4047	.3874	.3948	.3790	.3860	.3713	.3780	.3642	.3711	.3580
0.7	.3693	.3503	.3624	.3450	.3558	.3337	.3497	.3346	.3442	[illegible]
0.8	.3310	.3112	.3273	.3091	.3232	.3064	.3191	.3033	.3152	.3002
0.9	.2911	.2708	.2906	.2720	.2892	.2719	.2872	.2709	.2850	[illegible]
1.0	.2503	.2298	.2532	.2344	.2544	.2370	.2545	.2382	.2540	.2335
1.1	.2095	.1892	.2158	.1971	.2197	.2024	.2219	.2057	.2231	.2077
1.2	.1694	.1500	.1789	.1611	.1855	.1689	[illegible]	.1742	.1927	[illegible]
1.3	.1308	.1122	.1435	.1264	.1526	.1367	.1589	.1441	.1635	.1493
1.4	.0941	.0767	.1098	.0938	.1213	.1064	.1295	.1156	.1356	.1224
1.5	.0599	.0438	.0784	.0637	.0922	.0785	.1022	.0894	.1097	.0975
1.6	.0284	.0139	.0495	.0362	.0653	.0529	.0770	.0654	.0858	.0745
1.7	.0000	.0000	.0235	.0113	.0411	.0299	.0523	.0437	.0643	.0543
1.8			.0000	.0000	.0193	.0094	.0338	.0245	.0449	.0361
1.9					.0000	.0000	.0157	.0076	.0277	.0200
2.0							.0000	.0000	.0128	.0062
2.1									.0000	.0000
2.2										
2.3										
2.4										
2.5										
2.6										
2.7										
2.8										
2.9										

TABLE 1. PROBABILITY DENSITIES FOR THE $g_c(x)$-FUNCTION

	2.2		2.3		2.4		2.5		2.6	
x	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05
0.0	0.4453	0.4447	0.4368	0.4362	0.4298	0.4292	0.4237	0.4232	0.4189	0.4183
0.1	.4430	.4400	.4346	.4316	.4276	.4247	.4216	.4188	.4168	.4141
0.2	.4357	.4304	.4275	.4224	.4207	.4158	.4150	.4102	.4103	.4056
0.3	.4239	.4163	.4162	.4089	.4098	.4027	.4043	.3974	.3999	.3931
0.4	.4078	.3983	.4007	.3916	.3948	.3859	.3897	.3811	.3856	.3771
0.5	.3880	.3767	.3817	.3708	.3763	.3658	.3717	.3615	.3690	[illegible]
0.6	.3648	.3523	.3594	.3474	.3548	.3431	.3507	.3394	.3474	[illegible]
0.7	.3392	.3255	.3347	.3215	.3309	.3181	.3275	.3151	.3247	.3125
0.8	.3115	.2972	.3081	.2943	.3051	.2918	.3024	.2894	.3001	.2874
0.9	.2826	.2679	.2803	.2661	.2792	.2645	.2762	.2629	.2745	.2614
1.0	.2531	.2382	.2519	.2376	.2507	.2369	.2494	.2360	.2483	.2351
1.1	.2235	.2088	.2235	.2093	.2232	.2095	.2226	.2093	.2221	[illegible]
1.2	.1945	.1804	.1955	.1820	.1961	.1830	.1963	.1835	.1963	[illegible]
1.3	.1665	.1531	.1687	.1557	.1701	.1575	.1710	.1588	.1715	.1556
1.4	.1400	.1273	.1431	.1309	.1453	.1335	.1469	.1354	.1480	.1369
1.5	.1152	.1036	.1193	.1081	.1222	.1114	.1244	.1139	.1260	.1157
1.6	.0924	.0819	.0974	.0872	.1010	.0912	.1038	.0942	.1058	.0964
1.7	.0718	.0623	.0776	.0684	.0818	.0729	.0851	.0764	.0875	.0790
1.8	.0533	.0449	.0598	.0516	.0646	.0567	.0683	.0607	.0711	.0636
1.9	.0369	.0235	.0440	.0369	.0493	.0425	.0534	.0468	.0565	[illegible]
2.0	.0277	.0163	.0303	.0242	.0361	.0301	.0405	.0348	.0439	.0333
2.1	.0104	.0050	.0185	.0133	.0247	.0196	.0294	.0246	.0330	.0283
2.2	.0000	.0000	.0085	.0040	.0150	.0106	.0200	.0158	.0238	.0197
2.3			.0000	.0000	.0067	.0032	.0120	.0086	.0160	.0126
2.4					.0000	.0000	.0054	.0026	.0096	.0067
2.5							.0000	.0000	.0042	.0021
2.6									.0000	.0000
2.7										
2.8										

TABLE 1. PROBABILITY DENSITIES FOR THE $g_c(x)$-FUNCTION

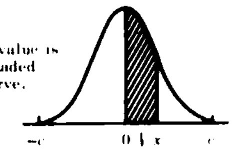
x	2.7		2.8		2.9		3.0	
	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05
0.0	0.4147	0.4142	0.4113	0.4108	0.4087	0.4082	0.4063	0.4058
0.1	.4127	.4100	.4093	.4067	.4067	.4041	.4043	.4013
0.2	.4063	.4017	.4030	.3985	.4005	.3960	.3981	.3937
0.3	.3960	.3894	.3929	.3864	[illegible]	.3840	.3883	.3819
0.4	.3820	.3737	.3791	.3700	.3755	.3657	[illegible]	.3667
0.5	.3647	.3549	.3621	.3524	.3600	.3504	.3591	[illegible]
0.6	.3446	.3337	.3422	.3315	.3403	.3297	.3386	.3291
0.7	.3223	.3103	.3202	.3084	[illegible]	.3069	.3171	.3056
0.8	.2981	.2850	.2965	.2841	.2951	[illegible]	.2933	.2818
0.9	.2729	.2601	.2716	.2590	.2705	.2581	.2695	.2572
1.0	.2472	.2343	.2463	.2335	.2455	.2329	.2447	.2322
1.1	.2215	.2087	.2209	.2083	.2204	.2079	[illegible]	.2075
1.2	.1962	.1832	.1960	.1833	[illegible]	.1830	.1955	.1836
1.3	.1719	.1591	.1720	.1604	.1720	[illegible]	.1720	[illegible]
1.4	.1497	.1377	.1492	.1383	.1495	.1393	.1496	.1399
1.5	.1271	.1170	.1279	.1178	[illegible]	.1186	.1298	.1191
1.6	.1073	.0931	.1034	.0993	.1001	.1002	.1097	[illegible]
1.7	.0893	.0810	.0937	[illegible]	[illegible]	[illegible]	.0924	[illegible]
1.8	.0732	.0659	.0740	.0675	.0752	.0683	[illegible]	.0697
1.9	.0589	.0525	.0607	.0544	[illegible]	.0552	.0630	[illegible]
2.0	.0465	.0410	.0495	.0430	.0490	.0445	.0511	.0457
2.1	.0359	.0312	.0380	.0333	.0395	.0349	.0408	.0363
2.2	.0268	.0227	.0290	.0250	.0307	.0267	.0320	[illegible]
2.3	.0191	.0159	.0215	.0182	.0232	.0209	.0240	.0214
2.4	.0128	.0100	.0153	.0125	.0171	.0144	.0195	.0155
2.5	.0076	.0054	.0101	.0080	.0120	.0099	.0135	.0114
2.6	.0034	.0016	.0060	.0042	.0079	.0061	.0095	.0077
2.7	.0000	.0000	.0026	.0013	.0046	.0032	.0062	.0048
2.8			.0000	.0000	.0020	.0009	.0036	.0026
					.0000	.0000	.0015	.0007
							.0000	.0000

TABLE 2. VARIANCES OF THE $g_c(x)$-FUNCTION

c	σ_g	c	σ_g^2	c	σ_g^2
0.80	0.125	2.00	0.611	3.00	0.918
0.90	.155	2.05	.632	3.05	.927
1.00	.190	2.10	.652	3.10	.934
1.10	.223	2.15	.671	3.15	.943
1.20	.264	2.20	.691	3.20	.947
1.25	.287	2.25	.711	3.25	.954
1.30	.305	2.30	.729	3.30	.959
1.35	.325	2.35	.748	3.35	.963
1.40	.346	2.40	.764	3.40	.968
1.45	.367	2.45	.782	3.45	.972
1.50	.391	2.50	.797	3.50	.974
1.55	.411	2.55	.812	3.55	.978
1.60	.434	2.60	.827	3.60	.981
1.65	.456	2.65	.841	3.65	.984
1.70	.479	2.70	.853	3.70	.986
1.75	.501	2.75	.866	3.75	.987
1.80	.523	2.80	.879	3.80	.989
1.85	.545	2.85	.888	3.85	.991
1.90	.567	2.90	.899	3.90	.992
1.95	.588	2.95	.909	3.95	.993
				4.00	.996

TABLE 3. THE INCOMPLETE $G_c(x)$-FUNCTION (CUMULATIVE PROBABILITIES OF THE $g_c(x)$-FUNCTION) (Continued through page 33)

A typical tabulated value is illustrated by the shaded portion under the curve.



c / x	1.2 0.00	1.2 0.05	1.3 0.00	1.3 0.05	1.4 0.00	1.4 0.05	1.5 0.00	1.5 0.05	1.6 0.00	1.6 0.05
0.0	0.0000	0.0336	0.0000	0.0314	0.0000	0.0296	0.0000	0.0281	0.0000	0.0268
0.1	.0671	.1003	.0628	.0939	.0592	.0886	.0562	.0841	.0536	.0802
0.2	.1332	.1651	.1248	.1548	.1177	.1462	.1117	.1388	.1067	.1325
0.3	.1964	.2266	.1843	.2129	.1741	.2013	.1654	.1914	.1580	.1830
0.4	.2559	.2838	.2407	.2674	.2278	.2534	.2168	.2413	.2073	.2310
0.5	.3108	.3358	.2933	.3175	.2782	.3017	.2652	.2879	.2541	.2760
0.6	.3598	.3818	.3408	.3625	.3243	.3456	.3099	.3307	.2974	.3177
0.7	.4019	.4206	.3626	.4015	.3654	.3843	.3502	.3689	.3368	.3552
0.8	.4370	.4519	.4185	.4341	.4015	.4176	.3861	.4022	.3723	.3885
0.9	.4646	.4755	.4481	.4603	.4321	.4453	.4171	.4308	.4035	.4174
1.0	.4843	.4912	.4710	.4799	.4570	.4673	.4432	.4543	.4302	.4419
1.1	.4961	.4990	.4872	.4925	.4762	.4836	.4642	.4728	.4525	.4619
1.2	.5000		.4969	.4994	.4896	.4944	.4802	.4866	.4702	.4776
1.3			.5000		.4975	.4995	.4914	.4952	.4837	.4888
1.4					.5000		.4978	.4995	.4929	.4961
1.5							.5000		.4983	.4995
1.6									.5000	

TABLE 3. THE INCOMPLETE $G_\nu(x)$-FUNCTION (CUMULATIVE PROBABILITIES OF THE $g_\nu(x)$-FUNCTION)

ν	1.7		1.8		1.9		2.0		2.1	
x	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05
0.0	0.0000	0.0257	0.0000	0.0248	0.0000	0.0240	0.0000	0.0233	0.0000	0.0227
0.1	.0514	.0770	.0495	.0742	.0479	.0718	.0466	.0697	.0454	.0680
0.2	.1024	.1272	.0986	.1226	.0955	.1187	.0927	.1154	.0905	.1125
0.3	.1518	.1758	.1463	.1695	.1417	.1642	.1377	.1596	.1343	.1558
0.4	.1993	.2221	.1923	.2144	.1863	.2078	.1812	.2021	.1768	.1973
0.5	.2445	.2658	.2361	.2568	.2289	.2491	.2227	.2425	.2175	.2368
0.6	.2866	.3064	.2771	.2964	.2689	.2878	.2618	.2804	.2558	.2741
0.7	.3252	.3433	.3148	.3326	.3059	.3234	.2981	.3153	.2915	.3084
0.8	.3602	.3763	.3493	.3653	.3398	.3556	.3316	.3471	.3245	.3399
0.9	.3913	.4053	.3802	.3943	.3705	.3845	.3619	.3758	.3545	.3684
1.0	.4184	.4304	.4074	.4196	.3977	.4099	.3890	.4013	.3814	.3938
1.1	.4414	.4514	.4309	.4412	.4214	.4319	.4128	.4235	.4053	.4161
1.2	.4603	.4684	.4506	.4592	.4416	.4506	.4334	.4426	.4261	.4355
1.3	.4754	.4814	.4668	.4735	.4585	.4658	.4509	.4584	.4439	.4518
1.4	.4865	.4909	.4793	.4845	.4722	.4780	.4652	.4714	.4588	.4654
1.5	.4943	.4968	.4888	.4923	.4829	.4871	.4768	.4816	.4711	.4763
1.6	.4987	.4997	.4952	.4973	.4907	.4937	.4858	.4893	.4809	.4849
1.7	.5000		.4988	.4996	.4960	.4977	.4923	.4947	.4883	.4913
1.8			.5000		.4991	.4997	.4968	.4981	.4938	.4958
1.9					.5000		.4992	.4997	.4974	.4986
2.0							.5000		.4995	.4999
2.1									.5000	

TABLE 3. THE INCOMPLETE $G_c(x)$-FUNCTION (CUMULATIVE PROBABILITIES OF THE $g_c(x)$-FUNCTION)

c	2.2		2.3		2.4		2.5		2.6	
x	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05
0.0	0.0000	0.0222	0.0000	0.0218	0.0000	0.0214	0.0000	0.0211	0.0000	0.0209
0.1	.0222	.0665	.0436	.0652	.0429	.0642	.0423	.0633	.0418	.0626
0.2	.0885	.1101	.0868	.1080	.0854	.1063	.0842	.1048	.0832	.1036
0.3	.1314	.1524	.1290	.1496	.1269	.1472	.1252	.1458	.1237	.1435
0.4	.1730	.1932	.1698	.1896	.1672	.1866	.1649	.1841	.1630	.1821
0.5	.2129	.2320	.2090	.2278	.2058	.2243	.2030	.2213	.2008	.2189
0.6	.2506	.2685	.2461	.2638	.2424	.2598	.2392	.2564	.2366	.2537
0.7	.2857	.3024	.2807	.2972	.2766	.2929	.2730	.2892	.2701	.2861
0.8	.3183	.3335	.3129	.3280	.3084	.3233	.3045	.3193	.3014	.3161
0.9	.3480	.3617	.3423	.3560	.3378	.3511	.3335	.3469	.3301	.3435
1.0	.3748	.3870	.3689	.3812	.3640	.3762	.3597	.3719	.3562	.3693
1.1	.3986	.4094	.3927	.4035	.3877	.3985	.3833	.3942	.3798	.3906
1.2	.4195	.4290	.4136	.4232	.4087	.4182	.4043	.4139	.4007	.4103
1.3	.4376	.4456	.4319	.4400	.4270	.4352	.4227	.4309	.4191	.4274
1.4	.4528	.4596	.4474	.4543	.4427	.4498	.4385	.4456	.4350	.4422
1.5	.4656	.4710	.4606	.4662	.4561	.4619	.4521	.4580	.4488	.4549
1.6	.4760	.4903	.4714	.4759	.4673	.4720	.4635	.4684	.4603	.4653
1.7	.4841	.4875	.4801	.4837	.4763	.4802	.4729	.4769	.4699	.4741
1.8	.4904	.4928	.4870	.4897	.4837	.4867	.4806	.4838	.4779	.4812
1.9	.4949	.4965	.4921	.4941	.4894	.4916	.4867	.4891	.4843	.4869
2.0	.4979	.4988	.4959	.4971	.4937	.4953	.4914	.4932	.4893	.4913
2.1	.4994	.4998	.4982	.4990	.4966	.4977	.4948	.4961	.4930	.4946
2.2	.5000		.4996	.4999	.4986	.4993	.4973	.4982	.4970	.4979
2.3			.5000		.4997	.4999	.4989	.4994	.4979	.4986
2.4					.5000		.4997	.5000	.4992	.4996
2.5							.5000		.4998	.5000
2.6									.5000	

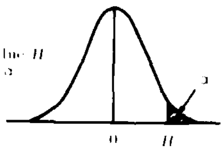
TABLE 3. THE INCOMPLETE $G_c(x)$-FUNCTION (CUMULATIVE PROBABILITIES OF THE $g_c(x)$-FUNCTION)

c	2.7		2.8		2.9		3.0	
x	0.00	0.05	0.00	0.05	0.00	0.05	0.00	0.05
0.0	0.0000	0.0207	0.0000	0.0205	0.0000	0.0201	0.0000	0.0203
0.1	.0414	.0620	.0410	.0615	.0408	.0611	.0405	.0607
0.2	.0824	.1026	.0818	.1018	.0312	.1011	.0808	.1005
0.3	.1225	.1421	.1215	.1410	.1208	.1401	.1201	.1393
0.4	.1614	.1803	.1602	.1789	.1591	.1778	.1582	.1767
0.5	.1389	.2168	.1973	.2151	.1961	.2139	.1950	.2125
0.6	.2344	.2513	.2326	.2494	.2311	.2479	.2298	.2465
0.7	.2676	.2835	.2656	.2814	.2640	.2797	.2625	.2782
0.8	.2986	.3132	.2964	.3110	.2947	.3091	.2931	.3075
0.9	.3272	.3404	.3249	.3381	.3230	.3362	.3213	.3344
1.0	.3532	.3652	.3507	.3627	.3488	.3607	.3470	.3589
1.1	.3766	.3874	.3741	.3848	.3721	.3829	.3702	.3809
1.2	.3975	.4071	.3949	.4045	.3929	.4024	.3910	.4005
1.3	.4160	.4243	.4134	.4217	.4113	.4196	.4094	.4177
1.4	.4319	.4392	.4294	.4366	.4273	.4346	.4254	.4327
1.5	.4458	.4518	.4433	.4494	.4412	.4474	.4393	.4455
1.6	.4575	.4626	.4551	.4602	.4531	.4583	.4512	.4565
1.7	.4672	.4715	.4650	.4693	.4631	.4674	.4613	.4657
1.8	.4754	.4788	.4733	.4768	.4715	.4750	.4698	.4734
1.9	.4820	.4847	.4800	.4829	.4784	.4813	.4768	.4797
2.0	.4873	.4894	.4855	.4877	.4840	.4863	.4825	.4848
2.1	.4913	.4930	.4897	.4915	.4884	.4902	.4870	.4889
2.2	.4944	.4957	.4931	.4945	.4919	.4933	.4906	.4922
2.3	.4968	.4976	.4956	.4966	.4946	.4956	.4935	.4946
2.4	.4983	.4989	.4974	.4982	.4966	.4974	.4956	.4965
2.5	.4993	.4996	.4987	.4991	.4980	.4985	.4972	.4973
2.6	.4998	.5000	.4995	.4998	.4990	.4994	.4983	.4988
2.7	.5000		.4999	.5000	.4996	.4993	.4991	.4994
2.8			.5000		.4999	.5000	.4996	.4998
2.9					.5000		.4998	.4999
3.0							.5000	

TABLE 4. PERCENTAGE POINTS OF THE $H(\varepsilon, \theta)$-STATISTIC (Continued through page 52)

$\varepsilon = 1.2$

A typical tabulated value H is illustrated for some α (shaded area).



θ \ α	0.10	0.05	0.02	0.01
1	3.2242	6.1585	14.2034	26.9461
2	1.4911	2.0856	3.1088	4.2013
3	1.2015	1.5423	2.0269	2.4725
4	1.0887	1.3455	1.6725	1.9489
5	1.0289	1.2461	1.5020	1.7067
6	0.9922	1.1860	1.4029	1.5692
7	.9677	1.1463	1.3382	1.4811
8	.9498	1.1177	1.2926	1.4202
9	.9360	1.0964	1.2592	1.3757
10	.9253	1.0799	1.2337	1.3414
11	.9171	1.0668	1.2132	1.3148
12	.9100	1.0561	1.1967	1.2932
13	.9043	1.0469	1.1826	1.2750
14	.8992	1.0397	1.1712	1.2602
15	.8951	1.0329	1.1614	1.2475
16	.8916	1.0276	1.1529	1.2365
17	.8885	1.0227	1.1458	1.2267
18	.8854	1.0183	1.1391	1.2183
19	.8829	1.0145	1.1333	1.2111
20	.8809	1.0111	1.1284	1.2043
21	.8788	1.0082	1.1239	1.1984
22	.8768	1.0053	1.1195	1.1933
23	.8752	1.0028	1.1159	1.1882
24	.8737	1.0004	1.1123	1.1840
25	.8722	0.9985	1.1092	1.1797
26	.8711	.9965	1.1065	1.1764
27	.8696	.9946	1.1038	1.1730
28	.8686	.9927	1.1012	1.1696
29	.8676	.9912	1.0989	1.1666
30	.8666	.9896	1.0969	1.1641

TABLE 4. PERCENTAGE POINTS OF THE $H(\nu; c, \delta)$-STATISTIC

$c = 1.3$

ν \ α	0.10	0.05	0.02	0.01
1	3.4928	6.6123	15.2959	29.1710
2	1.6153	2.2393	3.3460	4.5482
3	1.3016	1.6559	2.1828	2.6766
4	1.1794	1.4446	1.8011	2.1098
5	1.1147	1.3380	1.6175	1.8477
6	1.0748	1.2734	1.5103	1.6987
7	1.0483	1.2308	1.4411	1.6034
8	1.0289	1.2001	1.3921	1.5374
9	1.0140	1.1772	1.3560	1.4893
10	1.0024	1.1595	1.3296	1.4522
11	0.9935	1.1454	1.3065	1.4233
12	.9858	1.1340	1.2887	1.4000
13	.9797	1.1241	1.2738	1.3302
14	.9742	1.1163	1.2613	1.3642
15	.9697	1.1093	1.2507	1.3505
16	.9659	1.1033	1.2416	1.3386
17	.9626	1.0981	1.2339	1.3280
18	.9592	1.0934	1.2267	1.3188
19	.9565	1.0892	1.2205	1.3111
20	.9542	1.0856	1.2152	1.3037
21	.9520	1.0824	1.2104	1.2973
22	.9498	1.0793	1.2055	1.2918
23	.9482	1.0767	1.2017	1.2863
24	.9465	1.0741	1.1979	1.2817
25	.9448	1.0720	1.1945	1.2771
26	.9437	1.0700	1.1916	1.2735
27	.9421	1.0679	1.1887	1.2698
28	.9410	1.0658	1.1858	1.2662
29	.9399	1.0642	1.1834	1.2629
30	.9385	1.0625	1.1810	1.2602

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 1.4$

θ \ α	0.10	0.05	0.02	0.01
1	3.7231	7.0661	16.3885	31.1487
2	1.7218	2.3930	3.5871	4.8545
3	1.3875	1.7696	2.3387	2.8581
4	1.2572	1.5438	1.9296	2.2528
5	1.1882	1.4298	1.7330	1.9729
6	1.1457	1.3608	1.6187	1.8139
7	1.1174	1.3152	1.5440	1.7121
8	1.0968	1.2824	1.4915	1.6417
9	1.0808	1.2579	1.4529	1.5903
10	1.0685	1.2390	1.4235	1.5505
11	1.0590	1.2240	1.3998	1.5198
12	1.0508	1.2118	1.3808	1.4949
13	1.0443	1.2012	1.3648	1.4738
14	1.0384	1.9239	1.3514	1.4567
15	1.0337	1.1851	1.3401	1.4420
16	1.0296	1.1790	1.3303	1.4293
17	1.0260	1.1734	1.3221	1.4180
18	1.0225	1.1684	1.3143	1.4083
19	1.0195	1.1640	1.3076	1.3999
20	1.0172	1.1601	1.3020	1.3921
21	1.0148	1.1567	1.2958	1.3853
22	1.0124	1.1534	1.2917	1.3794
23	1.0107	1.1506	1.2876	1.3735
24	1.0089	1.1479	1.2834	1.3686
25	1.0071	1.1456	1.2798	1.3637
26	1.0060	1.1434	1.2767	1.3598
27	1.0042	1.1412	1.2736	1.3559
28	1.0030	1.1389	1.2708	1.3520
29	1.0018	1.1373	1.2680	1.3486
30	1.0007	1.1356	1.2654	1.3456

TABLE 4. PERCENTAGE POINTS OF THE $H(\varepsilon, \theta)$-STATISTIC

$\varepsilon = 1.5$

θ \ α	0.10	0.05	0.02	0.01
1	3.9534	7.5847	17.7542	33.3736
2	1.8283	2.5686	3.8860	5.2034
3	1.4733	1.8994	2.5336	3.0623
4	1.3349	1.6571	2.0906	2.4137
5	1.2617	1.5347	1.8775	2.1139
6	1.2166	1.4607	1.7536	1.9435
7	1.1865	1.4119	1.6727	1.8344
8	1.1646	1.3765	1.6158	1.7539
9	1.1477	1.3503	1.5739	1.7039
10	1.1346	1.3300	1.5421	1.6614
11	1.1245	1.3139	1.5165	1.6284
12	1.1158	1.3007	1.4958	1.6016
13	1.1089	1.2894	1.4785	1.5791
14	1.1026	1.2804	1.4640	1.5608
15	1.0976	1.2721	1.4519	1.5450
16	[illegible]	1.2655	1.4412	1.5314
17	1.0895	1.2595	1.4322	1.5193
18	1.0857	1.2542	1.4239	1.5088
19	1.0826	1.2494	1.4166	1.4999
20	1.0801	1.2452	1.4105	1.4916
21	1.0776	1.2416	1.4049	1.4842
22	1.0751	1.2380	1.3993	1.4779
23	1.0732	1.2351	1.3948	1.4716
24	1.0713	1.2321	1.3904	1.4664
25	1.0694	1.2297	1.3865	1.4611
26	1.0682	1.2273	1.3831	1.4570
27	1.0663	1.2249	1.3793	1.4528
28	1.0651	1.2225	1.3764	1.4486
29	1.0633	[illegible]	1.3730	1.4449
30	1.0626	1.2189	1.3703	1.4417

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 1.6$

θ \ α	0.10	0.05	0.02	0.01
1	4.1837	8.0385	18.9833	35.8457
2	1.9346	2.7223	4.1551	5.5988
3	1.5591	2.0131	2.7090	3.2891
4	1.4127	1.7562	2.2353	2.5025
5	1.3352	1.6266	2.0074	2.2704
6	1.2874	1.5481	1.8750	2.0874
7	1.2556	1.4962	1.7885	1.9703
8	1.2325	1.4589	1.7276	1.8392
9	1.2146	1.4311	1.6829	1.8301
10	1.2006	1.4096	1.6489	1.7845
11	1.1900	1.3925	1.6215	1.7490
12	1.1808	1.3786	1.5994	1.7203
13	1.1735	1.3665	1.5809	1.6961
14	1.1669	1.3570	1.5654	1.6764
15	1.1616	1.3482	1.5523	1.6595
16	1.1569	1.3412	1.5409	1.6448
17	1.1529	1.3349	1.5314	1.6319
18	1.1490	1.3292	1.5224	1.6206
19	1.1456	1.3241	1.5147	1.6110
20	1.1430	1.3197	1.5081	1.6020
21	1.1404	1.3159	1.5022	1.5942
22	1.1377	1.3121	1.4962	1.5874
23	1.1357	1.3090	1.4914	1.5806
24	1.1337	1.3058	1.4866	1.5750
25	1.1317	1.3033	1.4825	1.5694
26	1.1304	1.3007	1.4789	1.5649
27	1.1284	1.2982	1.4753	1.5604
28	1.1271	1.2957	1.4717	1.5559
29	1.1258	1.2938	1.4687	1.5519
30	1.1244	1.2919	1.4658	1.5485

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 1.7$

θ \ α	0.10	0.05	0.02	0.01
1	4.4140	8.4274	19.8028	38.0706
2	2.0413	2.8540	4.3344	5.9357
3	1.6450	2.1105	2.8259	3.4932
4	1.4904	1.3412	2.3318	2.7535
5	1.4087	1.7052	2.0941	2.4114
6	1.3583	1.6230	1.9559	2.2170
7	1.3248	1.5606	1.8657	2.0926
8	1.3003	1.5295	1.8022	2.0065
9	1.2814	1.5003	1.7556	1.9437
10	1.2667	1.4778	1.7201	1.8952
11	1.2556	1.4598	1.6914	1.8576
12	1.2458	1.4452	1.6684	1.8271
13	1.2381	1.4326	1.6491	1.8014
14	1.2311	1.4227	1.6330	1.7804
15	1.2255	1.4134	1.6193	1.7625
16	1.2206	1.4061	1.6074	1.7469
17	1.2164	1.3995	1.5975	1.7332
18	1.2122	1.3935	1.5882	1.7212
19	1.2087	1.3882	1.5801	1.7110
20	1.2059	1.3836	1.5732	1.7015
21	1.2031	1.3796	1.5670	1.6931
22	1.2003	1.37[illegible]	1.5608	1.6859
23	1.1982	1.3723	1.5558	1.6787
24	1.1961	1.3690	1.5508	1.6728
25	1.1940	1.3663	1.5464	1.6668
26	1.1926	1.3637	1.5427	1.6620
27	1.1905	1.3610	1.5390	1.6572
28	1.1891	1.3584	1.5352	1.6524
29	1.1878	1.3564	1.5321	1.6482
30	1.1864	1.3544	1.5290	1.6447

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 1.8$

θ \ α	0.10	0.05	0.02	0.01
1	4.6060	8.8164	20.7536	39.5539
2	2.1301	2.9858	4.5437	6.1670
3	1.7165	2.2079	2.9624	3.6294
4	1.5552	1.9262	2.4444	2.8607
5	1.4699	1.7840	2.1952	2.5053
6	1.4174	1.6979	2.0504	2.3034
7	1.3824	1.6410	1.9558	2.1741
8	1.3568	1.6001	1.8892	2.0847
9	1.3371	1.5696	1.8403	2.0194
10	1.3218	1.5460	1.3031	1.9691
11	1.3102	1.5272	1.7731	1.9299
12	1.2999	1.5120	1.7490	1.8982
13	1.2919	1.4988	1.7288	1.8715
14	1.2846	1.4884	1.7118	1.8498
15	1.2788	1.4786	1.6974	1.8311
16	1.2737	1.4710	1.6850	1.8150
17	1.2693	1.4641	1.6746	1.8007
18	1.2649	1.4578	1.6648	1.7883
19	1.2613	1.4523	1.6563	1.7777
20	1.2584	1.4474	1.6492	1.7678
21	1.2554	1.4433	1.6426	1.7591
22	1.2525	1.4391	1.6361	1.7515
23	1.2503	1.4356	1.6309	1.7442
24	1.2481	1.4322	1.6257	1.7379
25	1.2460	1.4294	1.6211	1.7317
26	1.2445	1.4266	1.6172	1.7268
27	1.2423	1.4238	1.6133	1.7219
28	1.2408	1.4211	1.6094	1.7168
29	1.2394	1.4190	1.6061	1.7125
30	1.2379	1.4160	1.6028	1.7087

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 1.9$

θ \ α	0.10	0.05	0.02	0.01
1	4.7979	9.2054	21.7148	41.5316
2	2.2188	3.1175	4.7529	6.4753
3	1.7880	2.3053	3.0988	3.8108
4	1.6201	2.0112	2.5570	3.0038
5	1.5312	1.8627	2.2963	2.6306
6	1.4764	1.7728	2.1446	2.4185
7	1.4400	1.7134	2.0459	2.2828
8	1.4134	1.6707	1.9762	2.1889
9	1.3928	1.6388	1.9251	2.1204
10	1.3769	1.6142	1.[illegible]862	2.0675
11	1.3647	1.5946	1.8548	2.0264
12	1.3541	1.5787	1.8295	1.9932
13	1.3457	1.5649	1.8034	1.9651
14	1.3361	1.5540	1.7906	1.9423
15	1.3321	1.5439	1.7756	1.9227
16	1.3267	1.5359	1.7625	1.9057
17	1.3222	1.5287	1.7517	1.8907
18	1.3176	1.5222	1.7415	1.8777
19	1.3138	1.5164	1.7326	1.8666
20	1.3108	1.5113	1.7251	1.8562
21	1.3078	1.5069	1.7183	1.8479
22	1.3047	1.5026	1.7115	1.8392
23	1.3024	1.4990	1.7060	1.8314
24	1.3002	1.4953	1.7005	1.9248
25	1.2979	1.4924	1.6958	1.9183
26	1.2964	1.4[illegible]	1.6917	1.8131
27	1.2941	[illegible]	1.6876	1.8079
28	1.2925	1.4838	1.6835	1.8026
29	1.2910	1.4816	1.6801	1.7981
30	1.2895	1.4794	1.6767	1.7942

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 2.0$

θ \ α	0.10	0.05	0.02	0.01
1	4.9514	9.5943	22.6708	43.5092
2	2.2899	3.2492	4.9622	6.7837
3	1.8452	2.4027	3.2352	3.9923
4	1.6719	2.0962	2.6695	3.1468
5	1.5802	1.9414	2.3974	2.7558
6	1.5237	1.8477	2.2392	2.5337
7	1.4860	1.7856	2.1359	2.3915
8	1.4586	1.7413	2.0632	2.2931
9	1.4374	1.7030	2.0098	2.2214
10	1.4210	1.6824	1.9692	2.1660
11	1.4084	1.6620	1.9364	2.1229
12	1.3974	1.6454	1.9101	2.0881
13	1.3888	1.6310	1.8880	2.0587
14	1.3810	1.6197	1.8694	2.0348
15	1.3747	1.6091	1.8538	2.0143
16	1.3692	1.6008	1.8402	1.9965
17	1.3645	1.5933	1.8288	1.9808
18	1.3598	1.5865	1.8182	1.9671
19	1.3559	1.5804	1.8089	1.9555
20	1.3527	1.5751	1.8011	1.9445
21	1.3496	1.5706	1.7939	1.9350
22	1.3465	1.5661	1.7868	1.9268
23	1.3441	1.5623	1.7811	1.9186
24	1.3418	1.5595	1.7754	1.9117
25	1.3394	1.5555	1.7704	1.9049
26	1.3373	1.5525	1.7662	1.8994
27	1.3355	1.5495	1.7619	1.8940
28	1.3339	1.5464	1.7576	1.8885
29	1.3323	1.5442	1.7540	1.8837
30	1.3308	1.5419	1.7505	1.8796

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 2.1$

θ \ α	0.10	0.05	0.02	0.10
1	5.1433	9.9833	23.6268	45.2397
2	2.3786	3.3809	5.1714	7.0535
3	1.9167	2.5001	3.3716	4.1511
4	1.7367	2.1811	2.7921	3.2720
5	1.6414	2.0201	2.4985	2.8655
6	1.5827	[illegible]	2.3336	2.6345
7	1.5436	1.8582	2.2260	2.4867
8	1.5151	1.8118	2.1502	2.3943
9	1.4931	1.7773	2.0946	2.3097
10	1.4760	1.7506	2.0522	2.2521
11	1.4630	1.7294	2.0181	2.2074
12	1.4516	1.7121	1.9906	2.1711
13	1.4426	1.6971	1.9676	2.1406
14	1.4345	1.6854	1.9483	2.1157
15	1.4280	1.6744	1.9320	2.0944
16	1.4223	1.6657	1.9178	2.0759
17	1.4174	1.6578	1.9060	2.0595
18	1.4125	1.6508	1.8948	2.0453
19	1.4084	1.6445	1.8852	2.0332
20	1.4052	1.6390	1.8770	2.0219
21	1.4019	1.6343	1.8696	2.0119
22	1.3986	1.6296	1.8622	2.0034
23	1.3962	1.6256	1.8562	1.9949
24	1.3938	1.6217	1.8503	1.9878
25	1.3913	1.6186	1.8451	1.9807
26	1.3897	1.6154	1.8406	1.9750
27	1.3872	1.6123	1.8362	1.9693
28	1.3856	1.6091	1.8317	1.9636
29	1.3840	1.6068	1.8280	1.9586
30	1.3824	1.6044	1.8243	1.9544

TABLE 4. PERCENTAGE POINTS OF THE $H(\cdot, c, \theta)$-STATISTIC

$c = 2.2$

θ \ α	0.10	0.05	0.02	0.01
1	5.2962	10.3074	24.4462	46.9702
2	2.4496	3.4907	5.3508	7.3233
3	1.9739	2.5813	3.4886	4.3099
4	1.7885	2.2520	2.8786	3.3971
5	1.6904	2.0856	2.5351	2.9751
6	1.6300	1.9851	2.4146	2.7353
7	1.5897	1.9185	2.3032	2.5818
8	1.5604	1.8707	2.2248	2.4755
9	1.5377	1.8350	2.1672	2.3980
10	1.5201	1.8074	2.1234	2.3383
11	1.5067	1.7855	2.0881	2.2918
12	1.4949	1.7676	2.0596	2.2542
13	1.4857	1.7522	2.0358	2.2224
14	1.4773	1.7401	2.0159	2.1966
15	1.4706	1.7287	1.9990	2.1745
16	1.4647	1.7198	1.9844	2.1553
17	1.4597	1.7117	1.9721	2.1383
18	1.4547	1.7044	1.9605	2.1236
19	1.4505	1.6979	1.9506	2.1110
20	1.4471	1.6922	1.9421	2.0992
21	1.4438	1.6873	1.9344	2.0889
22	1.4404	1.6825	1.9257	2.0800
23	1.4379	1.6784	1.9206	2.0712
24	1.4354	1.6744	1.9144	2.0638
25	1.4328	1.6711	1.9091	2.0564
26	1.4312	1.6679	1.9045	2.0505
27	1.4286	1.6646	1.8998	2.0449
28	1.4270	1.6614	1.8952	2.0387
29	[illegible]	1.6590	1.8914	2.0336
30	1.4236	1.6565	1.8876	2.0291

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 2.3$

θ \ α	0.10	0.05	0.02	0.01
1	5.4504	10.6316	25.4022	48.9479
2	2.5206	3.6005	5.5600	7.6316
3	2.0312	2.6625	3.6250	4.4913
4	1.8404	2.3228	2.9912	3.5402
5	1.7394	2.1512	2.6862	3.1003
6	1.6772	2.0475	2.5090	2.8504
7	1.6358	1.9789	2.3932	2.6905
8	1.6059	1.9295	2.3118	2.5708
9	1.5823	1.8927	2.2520	2.4990
10	1.5642	1.8642	2.2064	2.4367
11	1.5503	1.8416	2.1697	2.3883
12	1.5383	1.8232	2.1402	2.3491
13	1.5288	1.8073	2.1154	2.3160
14	1.5201	1.7948	2.0947	2.2891
15	1.5132	1.7831	2.0771	2.2660
16	1.5072	1.7739	2.0620	2.2460
17	1.5020	1.7655	2.0492	2.2284
18	1.4968	1.7580	2.0372	2.2130
19	1.4925	1.7513	2.0268	2.1999
20	1.4890	1.7454	2.0181	2.1976
21	1.4856	1.7404	2.0101	2.1768
22	1.4822	1.7354	2.0021	2.1676
23	1.4796	1.7312	1.9957	2.1594
24	1.4770	1.7270	1.9893	2.1507
25	1.4744	1.7237	1.9837	2.1430
26	1.4725	1.7203	1.9789	2.1369
27	1.4701	1.7170	1.9742	2.1307
28	1.4683	1.7136	1.9694	2.1246
29	1.4666	1.7111	[illegible]	2.11[illegible]
30	1.4649	1.7086	1.9614	2.1146

TABLE 4. PERCENTAGE POINTS OF THE $H(\varepsilon, \theta)$-STATISTIC

$\varepsilon = 2.4$

θ \ α	0.10	0.05	0.02	0.01
1	5.5655	10.8909	26.0850	50.4312
2	2.5739	3.6883	5.7095	7.3629
3	2.0741	2.7274	3.7224	4.6274
4	1.8793	2.3794	3.0716	3.6474
5	1.7761	2.2037	2.7584	3.1943
6	1.7127	2.0974	2.5764	2.9368
7	1.6704	2.0271	2.4576	2.7720
8	1.6395	1.97[illegible]	2.3740	2.6573
9	1.6157	1.9388	2.3125	2.5748
10	1.5972	1.9097	2.2658	2.5106
11	1.5831	1.8860	2.2281	2.4607
12	1.5708	1.8677	2.1977	2.4203
13	1.5611	1.8514	2.1723	2.3362
14	1.5522	1.8380	2.1510	2.3585
15	1.5452	1.8266	2.1330	2.3347
16	1.5390	1.8171	2.1174	2.3141
17	1.5337	1.8086	2.1043	2.2959
18	1.5284	1.8008	2.0920	2.2800
19	1.5240	1.7940	2.0813	2.2666
20	1.5205	1.7880	2.0723	2.2539
21	1.5170	1.782[illegible]	2.0641	2.2423
22	1.5135	1.7777	2.0559	2.2333
23	1.5108	1.7734	2.0494	2.2238
24	1.5082	1.7691	2.0428	2.2159
25	1.5055	1.7[illegible]	2.0371	2.2080
26	1.5038	1.7623	2.0321	2.2016
27	1.5011	1.7588	2.0272	2.1953
28	1.4994	1.7554	2.0223	2.1889
29	1.4970	1.752[illegible]	2.0182	2.1834
30	1.4958	1.7503	2.0141	2.1786

TABLE 4. PERCENTAGE POINTS OF THE $H(c, \theta)$-STATISTIC

$c = 2.5$

θ \ α	0.10	0.05	0.02	0.01
1	5.6807	11.2150	26.9044	52.1616
2	2.6271	3.7980	5.6889	8.1327
3	2.1170	2.8086	3.8394	4.7862
4	1.9192	2.4502	3.1581	3.7726
5	1.8129	2.2693	2.8451	3.3039
6	1.7481	2.1598	2.6574	3.0376
7	1.7049	2.0875	2.5343	2.8671
8	1.6734	2.0354	2.4485	2.7491
9	1.6491	1.9966	2.3851	2.6631
10	1.6302	1.9660	2.3369	2.5967
11	1.6158	1.9427	2.2980	2.5451
12	1.6032	1.9233	2.2668	2.5033
13	1.5934	1.9065	2.2406	2.4681
14	1.5844	1.8933	2.2186	2.4394
15	1.5772	1.8809	2.2000	2.4148
16	1.5709	1.8712	2.1839	2.3935
17	1.5655	1.8624	2.1704	2.3747
18	1.5601	1.8544	2.1577	2.3583
19	1.5556	1.8474	2.1467	2.3444
20	1.5520	1.8412	2.1374	2.3312
21	1.5484	1.8359	[illegible]	2.3198
22	1.5449	1.8306	2.1205	2.3099
23	1.5421	1.8262	2.1137	2.3001
24	1.5394	1.8219	2.1070	2.2919
25	1.5367	1.8183	2.1010	2.2837
26	1.5349	1.8147	2.0960	2.2772
27	1.5322	1.8112	2.0909	2.2706
28	1.5304	1.8077	2.0858	2.2640
29	1.5286	1.8050	2.0810	2.2583
30	1.5268	1.8024	2.0774	2.2534

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 2.6$

θ \ α	0.10	0.05	0.02	0.01
1	5.8342	11.4095	27.3604	53.6445
2	2.6981	3.8639	6.0981	8.3640
3	2.1742	2.8573	3.9758	4.9223
4	1.9700	2.4927	3.2806	3.8799
5	1.8619	2.3086	2.9462	3.3978
6	1.7951	2.1973	2.7518	3.1240
7	1.7510	2.1237	2.6248	2.9487
8	1.7187	2.0707	2.5356	2.[illegible]273
9	1.6937	2.0312	2.4699	2.7369
10	1.6743	2.0006	2.4200	2.6706
11	1.6595	1.9764	2.3797	2.6175
12	1.6466	1.9566	2.3473	2.5745
13	1.6364	1.9396	2.3202	2.5383
14	1.6272	1.9261	2.2974	2.5088
15	1.6198	1.9136	2.2781	2.4835
16	1.6133	1.9037	2.2615	2.4616
17	1.6078	1.8947	2.2475	2.4422
18	1.6022	1.8866	2.2344	2.4253
19	1.5976	1.8794	2.2230	2.4110
20	1.5939	1.8731	2.2134	2.3975
21	1.5902	1.8678	2.2049	2.3857
22	1.5865	1.8624	2.1959	2.3756
23	1.5838	1.8579	2.1888	2.3655
24	1.5810	1.8534	2.1818	2.3571
25	1.5782	1.8498	2.1757	2.3486
26	1.5764	1.8452	2.1704	2.3419
27	1.5736	1.8426	2.1652	2.3352
28	1.5717	1.8390	2.1599	2.3274
29	1.5699	1.8363	2.1556	2.3225
30	1.5680	1.8336	2.1512	2.3175

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 2.7$

	0.10	0.05	0.02	0.01
1	5.9110	11.6688	28.1336	54.3810
2	2.7336	3.9517	6.1579	8.5567
3	2.2028	2.9222	4.0148	5.0357
4	1.9959	2.5494	3.3128	3.9693
5	1.8864	2.3611	2.9751	3.4761
6	1.8190	2.2472	2.7788	3.1959
7	1.7740	2.1719	2.6506	3.0166
8	1.7413	2.1178	2.5604	2.8925
9	1.7160	2.0773	2.4941	2.8019
10	1.6963	2.0461	2.4437	2.7321
11	1.6814	2.0213	2.4030	2.6778
12	1.6682	2.0011	2.3703	2.6338
13	1.6580	1.9837	2.3429	2.5968
14	1.6486	1.9699	2.3199	2.5666
15	1.6411	1.9570	2.3005	2.5407
16	1.6346	1.9469	2.2837	2.5183
17	1.6289	1.9378	2.2695	2.4985
18	1.6233	1.9295	2.2563	2.4812
19	1.6186	1.9221	2.2448	2.4666
20	1.6149	1.9157	2.2350	2.4528
21	1.6111	1.9102	2.2262	2.4407
22	1.6074	1.9047	2.2174	2.4304
23	1.6046	1.9001	2.2103	2.4200
24	1.6018	1.8955	2.2032	2.4114
25	1.5990	1.8918	2.1970	2.4028
26	1.5971	1.8882	2.1917	2.3959
27	1.5943	1.8345	2.1864	2.3890
28	1.5924	1.8808	2.1811	2.3821
29	1.5906	1.8781	2.1767	2.3760
30	1.5887	1.8753	2.1723	2.3709

TABLE 4. PERCENTAGE POINTS OF THE $H(t, c, \theta)$-STATISTIC

$c = 2.8$

α	0.10	0.05	0.02	0.01
1	5.9877	11.7984	28.6799	56.1170
2	2.7691	3.9956	6.2775	8.7494
3	2.2314	2.9547	4.0927	5.1492
4	2.0218	2.5777	3.3771	4.0587
5	1.9109	2.3874	3.0328	3.5544
6	1.8426	2.2722	2.8327	3.2679
7	1.7971	2.1961	2.7021	3.0846
8	1.7639	2.1413	2.6101	2.9576
9	1.7383	2.1004	2.5425	2.8650
10	1.7184	2.0688	2.4912	2.7936
11	1.7032	2.0438	2.4497	2.7381
12	1.6899	2.0234	2.4164	2.6931
13	1.6795	2.0057	2.3884	2.6552
14	1.6700	1.9918	2.3650	2.6244
15	1.6624	1.9788	2.3452	2.5979
16	1.6558	1.9686	2.3280	2.5750
17	1.6501	1.9593	2.3136	2.5547
18	1.6444	1.9509	2.3001	2.5371
19	1.6396	1.9435	2.2884	2.5221
20	1.6359	1.9370	2.2784	2.5080
21	1.6321	1.9314	2.2694	2.4957
22	1.6283	1.9258	2.2604	2.4851
23	1.6254	1.9212	2.2532	2.4745
24	1.6226	1.9166	2.2460	2.4657
25	1.6197	1.9128	2.2397	2.4569
26	1.6178	1.9091	2.2343	2.4498
27	1.6150	1.9054	2.2289	2.4428
28	1.6131	1.9017	2.2235	2.4357
29	1.6112	1.8989	2.2190	2.4296
30	1.6093	1.8961	2.2145	2.4243

TABLE 4. PERCENTAGE POINTS OF THE $H(\alpha; c, \theta)$-STATISTIC

$c = 2.9$

θ \ α	0.10	0.05	0.02	0.01
1	6.0261	11.9929	29.3627	57.3531
2	2.7869	4.0615	6.4269	8.9421
3	2.2457	3.0034	4.1902	5.2626
4	2.0348	2.6202	3.4575	4.1481
5	1.9231	2.4267	3.1050	3.6327
6	1.8544	2.3097	2.9002	3.3399
7	1.8086	2.2323	2.7664	3.1525
8	1.7752	2.1766	2.6723	3.0228
9	1.7494	2.1350	2.6031	2.9282
10	1.7294	2.1030	2.5505	2.8552
11	1.7141	2.0775	2.5080	2.7984
12	1.7008	2.0567	2.4739	2.7525
13	1.6902	2.0388	2.4453	2.7137
14	1.6807	2.0246	2.4213	2.6822
15	1.6731	2.0114	2.4010	2.6552
16	1.6664	2.0010	2.3834	2.6317
17	1.6607	1.9916	2.3687	2.6110
18	1.6549	1.9831	2.3548	2.5930
19	1.6502	1.9755	2.3428	2.5777
20	1.6464	1.9689	2.3327	2.5633
21	1.6425	1.9633	2.3235	2.5506
22	1.6387	1.9576	2.3142	2.5398
23	1.6356	1.9529	2.3069	2.5290
24	1.6330	1.9482	2.2995	2.5200
25	1.6301	1.9444	2.2930	2.5113
26	1.6282	1.9406	2.2875	2.5038
27	1.6254	1.9369	2.2820	2.4966
28	1.6234	1.9331	2.2764	2.4894
29	1.6215	1.9302	2.2718	2.4831
30	1.6196	1.9274	2.2672	2.4777

TABLE 4. PERCENTAGE POINTS OF THE $H(\psi; c, \theta)$-STATISTIC

$c = 3.0$

θ \ α	0.10	0.05	0.02	0.01
1	6.1029	12.1226	29.7724	58.5[illegible]2
2	2.8224	4.1054	6.5166	9.1348
3	2.2743	3.0359	4.2487	5.3760
4	2.0607	2.6485	3.5058	4.2375
5	1.9476	2.4529	3.1484	3.7110
6	1.8780	2.3346	2.9407	3.4119
7	1.8316	2.2564	2.8050	3.2204
8	1.7978	2.2001	2.7096	3.0879
9	1.7717	2.1581	2.6394	2.9913
10	1.7514	2.1257	2.5861	2.9167
11	1.7360	2.0999	2.5430	2.8587
12	1.7224	2.0799	2.5084	2.8118
13	1.7118	2.0608	2.4794	2.7722
14	1.7021	2.0465	2.4551	2.7400
15	1.6944	2.0331	2.4345	2.7124
16	1.6876	2.0226	2.4167	2.6884
17	1.6818	2.0131	2.4017	2.6673
18	1.6760	2.0045	2.3877	2.6489
19	1.6712	1.9969	2.3755	2.6332
20	1.6673	1.9902	2.3652	2.6185
21	1.6634	1.9845	2.3559	2.6056
22	1.6596	1.9788	2.3465	2.5946
23	1.6567	1.9740	2.3390	2.5835
24	1.6539	1.9692	2.3316	2.5743
25	1.6509	1.9654	2.3250	2.5651
26	1.6490	1.9616	2.3194	2.5578
27	1.6461	1.9573	2.3138	2.5504
28	1.6441	1.9540	2.3082	2.5430
29	1.6422	1.9511	2.3035	2.5366
30	1.6403	1.9482	2.2988	2.5311

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Undersea Warfare Center San Diego, California 92152	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

TRANSFORMATIONS FOR STATISTICAL DISTRIBUTION APPROXIMATELY NORMAL BUT OF FINITE SAMPLE RANGE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research and Development — August 1965 to October 1966

5. AUTHOR(S) (First name, middle initial, last name)

R. H. Riffenburgh

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
30 October 1967	52	6

8a. CONTRACT OR GRANT NO. b. PROJECT NO. SR 104 03 01, Task 0586 (NEL L40571) c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) NUWC TP 19 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Ship Systems Command Department of the Navy

13. ABSTRACT

In many cases statistical data are drawn from a population with approximately normal distribution but with bounded, rather than infinite, domain. The traditional approach is to use a truncated normal distribution, but if the population probability approaches zero at the bounds of the domain, serious errors in hypothesis testing may accompany truncation, since the tails of the assumed distribution are used in error probability and critical region computation. In this report the truncated distribution is affine-transformed so that the abscissa is translated to the truncation points, and the curve above the new abscissa is given unit area; then the curve is half-rectified. The result is a quasi-normal distribution having finite domain yet retaining many properties of the normal distribution. Exact sampling theory, tests of hypothesis methodology, illustrative applications from electronic component reliability evaluation and ocean data analysis, and tables of associated probabilities are presented.

DD FORM 1473

UNCLASSIFIED
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14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Statistical Distributions						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

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